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ON THE ANALYSIS OF BALANCED INCOMPLETE MULTIRESPONSE DESIGNS

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Summary

The paper presents the analysis of a class of designs suitable for the experiments where p(>1) responses or characteristics are under study with only some and not all these characteristics being measured on each experimental unit. Such designs are called Incomplete Multiresponse (IM) designs. The general discussions of the issues involved in IM designs have been considered by Monahan [2] and Srivastava [6].

Key words : Variance-balanced designs, efficiency-balanced designs, C-matrix of block designs, Analysis of multiresponse designs.

Introduction

Let there be N experimental units divided into u disjoint sets S1, S2, . . . ,Su. ith set contains N_i units such that $\sum_{i=1}^{u} N_i = N$. Number of responses measured on each unit in S_i is p_i ($\leq$ p), with p_i less than p for at least one i, i = 1, 2, ...,u. The responses measured in the ith set are selected according to some rule D_1 which may be called "response-wise design". Let v denote the number of treatments under study which are same for all the sets. They give rise to p treatment effect vectors:-

$$\xi = \begin{bmatrix} \xi_{11} & \xi_{12} & \cdots & \xi_{1p} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \xi_{v1} & \xi_{v2} & \cdots & \xi_{vp} \end{bmatrix}$$

$$= \begin{bmatrix} \underset{\sim}{\xi}_1 & \underset{\sim}{\xi}_2 & \cdots & \underset{\sim}{\xi}_p \end{bmatrix}$$

A balanced connected block design D_{2i} is used in the ith set, i = 1,2,...,u. D_{2i} 's may be same or different in all the sets. For simplicity, Srivastava [5] took the same design D_{2i}, namely a BIBD in all the u sets, where u represents the different places or different points of time, where or when the designs D_{2i} 's are used. The reduced normal equations for the ith set are

$$E\,(Q_i)_{v \times p_i} = C_i\ \xi^{(i)}\ , \qquad i = 1, 2, \ldots, u$$

where C_i denotes C-matrix of D_{2i} and $\xi^{(i)}$ contains those p_i vectors of response measured in the ith set. Srivastava [5] used the following form

$$E(Q_i) = [C_i + a_i\, J_{v \times v}]\ \xi^{(i)}$$

where $\underline{J}\ \xi = \underline{O}$

His analysis depends on the decomposition of C_i in the form

$$C_i = \sum_{q=1}^{m} \alpha_{iq}\, F_q \qquad \text{such that} \qquad \sum_{q=1}^{m} F_q = J$$

α_{iq}'s are known scalars, J being a matrix with all unities and the proper choice of a_i's . Decomposition of C_i in the above form may not be always possible. Moreover his method of estimation of vectors of responses ξ is quite complicated.

It has been shown in this note that for any given response-wise design D_1 and for balanced connected (BC) block designs D_{2i} 's (D_{2i} 's may be same or different in the u sets), i = 1, ..., u, the analysis can be carried out much more easily in a straight forward manner. Further we show that, for efficiency balanced designs used in different sets, similar simpler analyses can also be achieved.

2. *Analysis for (variance) balanced designs*

Suppose p = Number of responses measured in the experiment;

u = Number of sets of experimental units.

Let p_i ($\leq p$) responses be measured on each experimental unit in the ith set, i = 1, 2, ... , u.

$$\text{Let} \quad \xi = \begin{bmatrix} \xi_{11} & \cdots & \xi_{1p} \\ \cdots & \cdots & \cdots \\ \xi_{v1} & \cdots & \xi_{vp} \end{bmatrix}_{v \times p} = [\ \xi_1\ \ \xi_2\ \ \cdots\cdots\ \ \xi_p]$$

= Expected values of the responses to different treatments.

Under the usual model

$$E(\underline{Q}_{i1}) = C_i\ \xi_1 \quad \text{where} \quad \underline{Q}_{i1} = \begin{bmatrix} Q_{i11} \\ Q_{i21} \\ \vdots \\ Q_{iv1} \end{bmatrix}_{v \times 1}$$

or $\quad E(Q_i) = Q_i\ \xi\ B^{(i)}$

where $\quad Q_i = [\ \underline{Q}_{i1} \ldots \underline{Q}_{ip_i}\ ]_{v \times p_i}$ and

$B^{(i)}$ is a $p \times p_i$ matrix where the column vector corresponding to response '1' measured in the ith set has unity in row '1' and zeroes elsewhere; i = 1, 2, ...,p. Now for a balanced connected design $C_i = \theta_i\ (I - v^{-1} J)$ where θ_i is the common positive characteristic root of the matrix C_i of D_{2i} used in the ith set $i = 1, \ldots, u$.

$$\therefore \quad E(Q_i) = \theta_i\ (I - v^{-1} J)\ \xi\ B^{(i)}$$

$$= \theta_i\ \xi\ B^{(i)}\ [\ \because\ \underline{1}'\ \xi = \underline{0}'\ ]$$

$$\Rightarrow \quad E(Q_1 \ldots Q_u) = \xi\ (\theta_1\ B^{(1)} \ldots \theta_u\ B^{(u)})$$

$$\therefore \text{ or } \quad E(Q) = \xi\ L \qquad (1)$$

where $\quad Q = (Q_1 \ldots Q_u)_{v \times \sum_i p_i}$ and $L = (\theta_1 B^{(1)} \ldots \theta_u B^{(u)})_{p \times \sum_i p_i}$

Now, (1) gives,

$$E(Q\ L') = \xi\ LL'. \qquad (2)$$

$$\text{Again,} \quad LL' = \sum_{i=1}^{n} \theta_i^2\ B^{(i)} B'^{(i)} = \sum_i \theta_i^2\ D_i$$

where D_i is a p xp diagonal matrix with unity at those p_i places which correspond to the pi responses studied on the ith set $i = 1, 2, \ldots, u$.

$$LL' = \sum_i \theta_i^2\ D_i = D \left(\sum_i \theta_i^2\ \delta_{i1} \ldots \sum_i \theta_i^2\ \delta_{ip} \right)$$

where $\delta_{i1} = 1$ if 1th response is measured in the ith set ;

$= O$ otherwise.

So $LL' = D(K_1 \ldots K_p)$ (say) .

Hence $R(LL') = p$

$\Rightarrow$ LL' is non-singular.

From (2)

$$E(Q L' (LL')^{-1}) = \xi$$

Now, $$Q L' = \sum_i \theta_i Q_i B^{(i)}$$

$$= \left(\sum_i (\theta_i \underline{Q}_{i1} \theta_{i1}) \ \ldots \ \sum_i (\theta_i \underline{Q}_{ip} \delta_{ip}) \right)$$

$\therefore$ $$(Q L')(LL')^{-1} = (\sum_i \theta_i \underline{Q}_{i1} \delta_{i1}/_{K_1} \ \ldots \ \sum_i \theta_i \underline{Q}_{ip} \delta_{ip}/_{K_p})$$

$$= (Z) \text{ (say)}$$

$\therefore$ $E(Z) = \xi$

or $$E \begin{bmatrix} \underline{Z}_1' \\ \cdot \\ \cdot \\ \underline{Z}_v' \end{bmatrix} = \xi \quad (3)$$

where $\underline{Z}_j' = (Z_{j1} \ \ldots \ Z_{jp})$, $j = 1, \ldots, v$ and $Z_{j1} = \sum_i \theta_i \frac{Q_{ij1} \ \delta_{i1}}{K_1}$

One can now easily check that the least square estimator of ξ is given by $\hat{\xi} = Z$. Here the rows of Z are linear functions of Q_{ij1}'s .

Further, $$\sum_{j=1}^{v} \underline{Z}_j' = \underline{O}'$$

This means not all Z_j's are linearly independent. Hence the dispersion matrix of Z will be singular.

One can easily verify that the pseudo dispersion matrix of Z comes out to be

$$D(Z)_{vp\times vp} = \frac{1}{v}\begin{bmatrix} \overline{v-1} & -1 & -1 & \dots & -1 \\ & \overline{v-1} & -1 & \dots & -1 \\ & & & & \\ & & & & \overline{v-1} \end{bmatrix} \times \Sigma^{*}_{p\times p}$$

where

$$\Sigma^{*}_{p\times p} = \begin{bmatrix} \frac{\sigma_{11}}{K_1^2}\sum_i \theta_i^3\,\delta_{i1}^2 & \frac{\sigma_{12}}{K_1K_2}\sum_i \theta_i^3\,\delta_{i1}\,\delta_{i2} & \dots & \frac{\sigma_{1p}}{K_1K_p}\sum_i \theta_i^3\,\delta_{i1}\,\delta_{ip} \\ & \frac{\sigma_{22}}{K_2^2}\sum_i \theta_i^3\,\delta_{i2}^2 & \dots & \frac{\sigma_{2p}}{K_2K_p}\sum_i \theta_i^3\,\delta_{i2}\,\delta_{ip} \\ & & & \vdots \\ & & & \frac{\sigma_{pp}}{K_p^2}\sum_i \theta_i^3\,\delta_{ip}^2 \end{bmatrix}$$

The rest of the analysis can be carried out on exactly the same lines as in Roy *et al.* [3].

3. *Analysis for efficiency balanced designs*

A connected block design is called efficiency balanced efficiency on

$\underline{1}'\,\hat{\underline{\tau}} = \dfrac{V(\underline{1}'\,\hat{\underline{\tau}})}{(\underline{1}'\,r^{-\delta}\underline{1})\,\sigma^2}$ is the same for all $\underline{1}' \,\varepsilon$ row space of the C-matrix. It is known [see, for example, Sinha and Saha [4] that a connected block design is EB if $C = \rho\,(r^{\delta} - rr'/n)$. Thus, in the present set up we have

$C_i = \rho_i\,(r_i^{\delta} - r_i\ r_i'/n)$ $\quad i = 1, 2, \dots, u$ where the symbols have their usual meanings.

Following the same procedure as in the previous section, we develop now the estimators for the efficiency balanced (EB) designs as follows:

$$E(Q_i) = C_i\,\xi\,B^{(i)}$$

$$= \rho_i\,(r_i^{\delta} - r_i\,r_i'/n)\,\xi\,B^{(i)} = \rho_i\,r_i^{\delta}\,\xi\,B^{(i)}$$

assuming $r_i' \xi = O'$

or, $E(r_i^{-\delta} Q_i) = \xi \rho_i B^{(i)}$

or, $E(r_1^{-\delta} Q_1 \quad \ldots \quad r_u^{-\delta} Q_u) = \xi L$

where $L = (\rho_1 B^{(1)} \quad \ldots \quad \rho_u B^{(u)})$

Now $L L' = \sum_i \rho_i^2 B^{(i)} B'^{(i)} = \sum_i \rho_i^2 D_i$

$$= \left(\sum_i \rho_i^2 \delta_{i1} \quad \ldots \quad \sum_i \rho_i^2 \delta_{ip} \right)$$

$$= D (K_1^* \quad \ldots \quad K_p^*) \quad \text{(say)}$$

where $\delta_{i1} = 1$ if lth response is measured in the ith set

$= 0$ otherwise, $i = 1, 2, \ldots, u$.

$\therefore$ $L L'$ is non-singular.

$\therefore$ $E(r_1^{-\delta} Q_1 \quad \ldots \quad r_u^{-\delta} Q_u) \; L' \; (L L')^{-1} = \xi$

Further,

$$(r_1^{-\delta} Q_1 \quad \ldots \quad r_u^{-\delta} Q_u) \; L' \; (L L')^{-1}$$

$$= \left(\sum_i (\rho_i r_i^{-\delta} \underline{Q}_{i1} \delta_{i1})/K_1^* \quad \ldots \quad \sum_i (\rho_i r_i^{-\delta} \underline{Q}_{ip} \delta_{ip})/K_p^* \right)$$

$$= Z \quad \text{(say)}$$

$\therefore$ $E(Z) = \xi$

or
$$E \begin{bmatrix} \underline{Z}_1' \\ \cdot \\ \cdot \\ \underline{Z}_v' \end{bmatrix} = \xi \tag{4}$$

where $Z_j' = (\underline{Z}_{j1} \; . \; . \; . \; \underline{Z}_{jp})$ and $\underline{Z}_{j1} = \sum_i \rho_i \, Q_{ij1} \, \delta_{i1}/r_{ij} \, K_1^*$

$j = 1, \ldots, v \, ; \; 1 = 1, 2, \; . \; . \; . \; , p \; .$

Here also, one can easily verify from (4) that the least square estimator of ξ is $\hat{\xi} = Z$. Since all $Z_j'^s$ are not linearly independent, the dispersion matrix of Z will be singular. One can easily show that the pseudo dispersion matrix of Z is given by

$$D(Z) = \begin{bmatrix} A_1 & B & B \ldots & B \\ B & A_2 & B \ldots & B \\ . & . & . \; . \; . \; . & . \\ B & B & B \ldots & A_v \end{bmatrix}_{vp \times vp}$$

where

$$A_j = \begin{bmatrix} \frac{\sigma_{11}}{K_1^{*2}} \sum_{i=1}^{u} \rho_i^3 \left(\frac{1}{r_{ij}} - \frac{1}{n}\right) \delta_{i1}^2 & \frac{\sigma_{12}}{K_1^* K_2^*} \sum_{i=1}^{u} \rho_i^3 \left(\frac{1}{r_{ij}} - \frac{1}{n}\right) \delta_{i1} \delta_{i2} & \cdots & \frac{\sigma_{1p}}{K_1^* K_p^*} \sum_{i=1}^{u} \rho_i^3 \left(\frac{1}{r_{ij}} - \frac{1}{n}\right) \delta_{i1} \delta_{ip} \\ & \frac{\sigma_{22}}{K_2^{*2}} \sum_{i=1}^{u} \rho_i^3 \left(\frac{1}{r_{ij}} - \frac{1}{n}\right) \delta_{i2}^2 & \cdots & \frac{\sigma_{2p}}{K_2^* K_p^*} \sum_{i=1}^{u} \rho_i^3 \left(\frac{1}{r_{ij}} - \frac{1}{n}\right) \delta_{i2} \delta_{ip} \\ & & & \\ & & & \frac{\sigma_{pp}}{K_p^{*2}} \sum_{i=1}^{u} \rho_i^3 \left(\frac{1}{r_{ij}} - \frac{1}{n}\right) \delta_{ip}^2 \end{bmatrix}_{p \times p}$$

and

$$B = \begin{bmatrix} -\frac{\sigma_{11}}{n} \frac{1}{K_1^{*2}} \sum_{i=1}^{u} \rho_i^3 \delta_{i1}^2 & -\frac{1}{n} \frac{\sigma_{12}}{K_1^* K_2^*} \sum_{i=1}^{u} \rho_i^3 \delta_{i1} \delta_{i2} & \cdots & -\frac{1}{n} \frac{\sigma_{1p}}{K_1^* K_p^*} \sum_{i=1}^{u} \rho_i^3 \delta_{i1} \delta_{ip} \\ & -\frac{1}{n} \frac{\sigma_{22}}{K_2^{*2}} \sum_{i=1}^{u} \rho_i^3 \delta_{i2}^2 & \cdots & -\frac{1}{n} \frac{\sigma_{2p}}{K_2^* K_p^*} \sum_{i=1}^{u} \rho_i^3 \delta_{i2} \delta_{ip} \\ & & & \vdots \\ & & & -\frac{1}{n} \frac{\sigma_{pp}}{K_p^{*2}} \sum_{i=1}^{u} \rho_i^3 \delta_{ip}^2 \end{bmatrix}_{p \times p}$$

The rest of the analysis can now be completed as indicated in Roy *et al.* [3]

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